ISRAEL JOURNAL OF MATHEMATICS, Vol. 58, No. 2, 1987

ON PHANTOM MAPS AND A THEOREM OF H. MILLER

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ABSTRACT

A map $f: X \to Y$ is a phantom map if any composition of f with a map from a finite complex into X is null homotopic. The proof of the Sullivan conjecture by H. Miller enables us to understand more deeply this phenomena. We prove, among other things, that any map from a space with finitely many non-vanishing homotopy groups into a finite complex is phantom and that any fibration over a 2-connected space with finitely many non-vanishing homotopy groups and with fiber a finite complex is trivial over each skeleton of the base.

This is a revised version of the first unpublished draft of the same title. The first draft was written in the midst of an exciting period when the Sullivan conjecture had just been proved by Haynes Miller and a great influx of applications were coming in.

Some of the results appearing in the first draft were omitted here as more complete versions have since been obtained (e.g. [6], [4]).

The excitement of the early days of the Sullivan conjecture proved completely justified by the size and volume of major developments it brought about. The past few years since the first draft was written did not diminish the author's gratitude to Haynes Miller for his first-hand information of the new results when still brewing, for many enlightening discussions and for a lot of encouragement.

My thanks are still due to W. Meier for his remarks and clarifications about phantom maps and for his proof of Theorem B(b) which greatly improved our original version. I am also grateful to E. Dror for his tutorials on completion and localization theories.

† Prof. Zabrodsky died in an automobile accident on November 20, 1986.

Received May 28, 1986

1. Basic notions and summary of results

A map $f\colon X \to Y$ between topological spaces is called a phantom map if for every map $h\colon K \to X$ from a finite polyhedron into X, $f \circ h$ is null homotopic. A null homotopic map is of course a phantom map, though a trivial one. Obviously, phantom maps cannot be distinguished from the constant maps by many of the standard homotopy functors.

As all our basic spaces are assumed to be connected countable CW complexes of finite type, a map $f\colon X \to Y$ is a phantom map if and only if its restriction to every skeleton of X is null homotopic.

The first example of a phantom map, as far as we know, is due to Adams and Walker [1]. A more familiar example is due to Gray [5].

The most thorough study of phantom maps was done by Meir [7–9] and some of our results are extensions of his observations. He should also be credited for realizing the strong relevancy of the Sullivan conjecture to phantom maps some eight years before it was actually resolved.

1.1. *Conventions and Notations*

We fix a set of primes $\mathbf{P}_1$ and consider all spaces to be $\mathbf{P}_1$ local. By a CW complex of finite type, we mean a CW-complex X with $\pi_1(X)$ finite and $\pi_m(X)$ finitely generated $Z_{\mathbf{P}_1}$ modules. All our basic spaces are assumed to be of finite type.

We use the following notations:

map(X, Y) — the space of continuous functions $X \to Y$,

map$_*(X, Y)$ — the subspace of pointed maps $X, x_0 \to Y, y_0$,

$\tilde{\theta}(X, Y)$ — the subspace of phantom maps in map$_*(X, Y)$,

$\theta(X, Y) = \pi_0(\tilde{\theta}(X, Y))$ — the homotopy classes of phantom maps $X \to Y$,

$\hat{C}_0(X, Y)$ — the path component of the constant map in map(X, Y),

$C_0(X, Y)$ — the path component of the constant map in map$_*(X, Y)$,

$\approx_w$ — a weak homotopy equivalence.

1.2. *Bousfield–Kan Rationalization*

If a CW complex X has a locally finite fundamental group, then X has a good rationalization $l_Q\colon X \to X_Q$ with the following properties:

(a) X_Q is simply connected, $\tilde{H}_i(X_Q, Z_{\mathbf{P}_1}) \approx \tilde{H}_i(X_Q, Q)$ and $\tilde{H}_i(l_Q, Z_{\mathbf{P}_1})$ is the tensoring with Q morphism.

(b) Let $\tau\colon X_\tau \to X$ be the homotopy fiber of l_Q. Then $X_\tau \xrightarrow{\tau} X \to X_Q$ is equivalent to a cofibration sequence.

(c) If $H^i(X, Z_{\mathbf{P}_1})$ contains no infinitely $\mathbf{P}_1$ divisible elements (as in the case of our ground condition that X is of finite type), then $H^i(\tau, Z_{\mathbf{P}_1})$ is a monomorphism.

(d) Any map $X \to Y_Q$, where Y_Q is a nilpotent Q-local space, factors uniquely up to homotopy through $l_Q: X \to X_Q$.

(e) If $j: Y \to Y_0$ is a rational equivalence between nilpotent spaces, then $\text{map}_*(X_Q, Y) \to \text{map}_*(X_Q, Y_0)$ is a weak homotopy equivalence.

1.3. *Description of Main Results*

In Section 2 we study phantom maps in general. We make the observation (Theorem B(b)) that a map $f: X \to Y$ is a phantom map if and only if $f \circ \tau \sim *$ (τ as in 1.2(b)) or equivalently if and only if f factors through the rationalization of X.

For a target space Y one can find a rational equivalence $j: Y \to Y_0$ so that $\Omega Y_0 \approx \prod_{m=2}^{\infty} K(\pi_m(Y)/\text{torsion}, m-1)$ ("integral approximation" as in Lemma A). Y_0 cannot be a target of a non-trivial phantom map (Theorem B(a), (b)). Thus one can describe the following procedure to obtain phantom maps (hidden in Theorem B): Given spaces X, Y (and one may assume without loss of generality that Y is simply connected) form the integral approximation $j: Y \to Y_0$ and let $h: \hat{Y} \to Y$ be the homotopy fiber of j, and $i: \Omega Y_0 \to \hat{Y}$ the usual map into the fibre. Choose a sequence of elements in

$$H^m(X_Q, \pi_m(Y)/\text{torsion}) \approx \text{Ext}(H_{m-1}(X, Q), \pi_m(Y)/\text{torsion}),$$

pull these elements back to

$$\Pi_m H^{m-1}(X_\tau, \pi_m(Y)/\text{torsion}) \to \Pi_m H^m(X_Q, \pi_m(Y)/\text{torsion})$$

(see 1.2(b)) thus obtaining a map $\hat{f}: X_\tau \to \Omega Y_0$. Compose $\hat{f}$ with i, $i \circ f: X_\tau \to \hat{Y}$; as $\pi_n(\hat{Y})$ are finite and as $H^*(X_\tau, M) \xrightarrow{\approx} H^*(X, M)$ for any finite group of coefficients, $[X, \hat{Y}] \to [X_\tau, \hat{Y}]$ is bijective and there exists a unique $\hat{f}_1 \in [X, \hat{Y}]$ so that $\hat{f}_1 \circ \tau \sim i \circ \hat{f}$.

The map $h \circ \hat{f}_1: X \to Y$ is a phantom map and every phantom map is obtained that way. This gives another explanation to the fact (see [8]) that $\theta(X, Y)$ is a quotient group of $A_0 = \Pi H^m(X_Q, \pi_m(Y)/\text{torsion})$.

Moreover, one can see that in the above procedure two elements in A_0 yield homotopic phantom maps $X \to Y$ if and only if they differ by an element in the image of the composition $[X_\tau, \Omega Y] \to [X_\tau, \Omega Y_0] \to A_0$. This is the essence of Theorem B, section 2.

In Section 3 we study the theorem of Miller ([11, Theorem A, "The Sullivan Conjecture") and observe that one can easily extend its validity to other function spaces extending both sources and targets. We use the following:

1.3.1. NOTATIONS. Let $\mathcal{K}$ denote the minimal class of homotopy types of CW complexes containing (the homotopy types of) finite dimensional complexes which satisfy:

(a) If $K \in \mathcal{K}$ and Y is any space then all path components of $\operatorname{map}(Y, K)$ and of $\operatorname{map}_*(Y, K)$ are in $\mathcal{K}$.

(b) If $F \to E \to B$ is a fibration and $B \in \mathcal{K}$ then $F \in \mathcal{K}$ if and only if $E \in \mathcal{K}$.

Let $\mathcal{K}_0$ denote the class of (homotopy types of) spaces K so that $\Omega K \in \mathcal{K}$. (Obviously $\mathcal{K} \subset \mathcal{K}_0$.)

Then we see (Corollary C′): If X is a space so that $\pi_n(X)$ are locally finite and $\pi_n(X) = 0$ for $n \geqq N$ then: For every $K \in \mathcal{K}$ $\operatorname{map}_*(X, K) \approx_w *$. For every $K_0 \in \mathcal{K}_0$, $C_0(X, K_0) \approx_w *$ and if X is simply connected then $\operatorname{map}_*(X, K_0) \approx_w *$. If X is simply connected and $K \in \mathcal{K}$ then $\operatorname{map}_*(X, B\operatorname{aut} K) \approx_w *$.

In Section 4 we apply Theorem C and Corollary C′ to study phantom maps. We prove (Theorem D) that all maps from a space X with finitely many non-vanishing homotopy groups into a finite complex (or into a space in $\mathcal{K}$) is a phantom map. The function space could be described completely in case the target is a rational H-space. Moreover, every fibration over a two connected space with finitely many non-vanishing homotopy groups whose fiber is a finite complex is trivial over each skeleton of the base.

In Section 5 we give some examples and applications of Theorem D.

1.4. *Precise Statements of the Main Theorems*

1.4.1. DEFINITIONS. Let Y be a CW complex of finite type. An integral approximation of Y is a space Y_0 with a map j: $Y \to Y_0$ so that: (a) j is a rational equivalence; (b) $\Omega Y_0 \approx \prod_{m=2}^{\infty} K(\pi_m(Y)/\text{torsion}, m-1)$. If a space Y_0 satisfies (b) (with $Y = Y_0$) we shall call it an integral approximation (without relating it to another space).

LEMMA A. (a) *Every simply connected CW complex of finite type has an integral approximation.*

(b) *If h: $X_1 \to X_2$ induces a monomorphism $H^i(h, Z_{P_1})$, $i \geqq 2$ and Y_0 is an integral approximation, then f: $X_2 \to Y_0$ is null homotopic if and only if $f \circ h \sim *$. (In function space terminology: if h^*: $\operatorname{map}_*(X_2, Y_0) \to \operatorname{map}_*(X_1, Y_0)$ is the map induced by h, then $(h^*)^{-1} C_0(X_1, Y_0) = C_0(X_2, Y_0)$.)*

The proof of Lemma A is given in Section 2. The main theorem of that section is the following:

THEOREM B. *Let X, Y be CW complexes of finite type, suppose Y is simply connected and* $j: Y \to Y_0$ *is an integral approximation. Consider the maps of function spaces induced by j and* $\tau: X_\tau \to X$:

$$\text{(DB)} \qquad \begin{array}{ccc} \text{map}_*(X, Y) & \xrightarrow{\tau^*} & \text{map}_*(X_\tau, Y) \\ \downarrow j_* & & \downarrow j_*^\tau \\ \text{map}_*(X, Y_0) & \xrightarrow{\tau_0^*} & \text{map}_*(X_\tau, Y_0) \end{array}$$

Then:

(a) *A map* $f_0: X \to Y_0$ *is null homotopic if and only if* $f_0 \circ \tau$ *is null homotopic, i.e.,* $(\tau_0^*)^{-1} C_0(X_\tau, Y_0) = C_0(X, Y_0)$.

(b) $f: X \to Y$ *is a phantom map if and only if* $f \circ \tau \sim *$ *(or equivalently if and only if it factors through the rationalization of X,* $f \sim f_0 \circ l_Q$*), i.e.,* $\tilde{\theta}(X, Y) = (\tau^*)^{-1} C_0(X_\tau, Y)$.

(c) *Clauses* (a) *and* (b) *and diagram* (DB) *induce a commutative diagram*:

$$\text{(DB)}_0 \qquad \begin{array}{ccc} \tilde{\theta}(X, Y) & \xrightarrow{\tau^*} & C_0(X_\tau, Y) \\ \downarrow j_* & & \downarrow j_*^\tau \\ C_0(X, Y_0) & \xrightarrow{\tau_0^*} & C_0(X_\tau, Y_0) \end{array}$$

Then $(\text{DB})_0$ *is a homotopy pull back diagram.*

(d) $\theta(X, Y)$ *is a divisible abelian group isomorphic to a quotient of* $\Pi_{i\geq 2}\text{Ext}(H_{i-1}(X, Q), \pi_i(Y)/\text{torsion})$.

In Section 3 we discuss some extensions of the following:

1.4.2. MILLER'S THEOREM ("The Sullivan conjecture", [11] theorem A). *Let G be a locally finite group, let K be a finite dimensional CW complex. Then* $\text{map}_*(BG, K) \approx_w *$.

To extend the domain and the range of the above theorem (and see [11] theorem A′) we use the following:

1.4.3. NOTATION. Let X be a CW complex. Denote by $M(X)$ the class of CW complex K satisfying $\text{map}_*(X, K) \approx_w *$. Let $M_0(X)$ denote the class of CW complexes K with $C_0(X, K) \approx_w *$. Obviously $M(X) \subset M_0(X)$.

We then prove:

THEOREM C. (a) *Let X be a simply connected space. Then $M_0(\Omega X) \subset M(X)$.*

(b) *If $F \to E \to B$ is a fibration then*

$$M(F) \cap M(B) = M(F) \cap M(E),$$

$$M_0(F) \cap M_0(B) = M_0(F) \cap M_0(E).$$

(c) *If $K \in M(X)$ ($K \in M_0(X)$) and f: $Y \to K$ is any map into K, then*

$$\mathrm{map}(Y, K)_f,\ \mathrm{map}_*(Y, K)_f \in M(X)$$

$$(\mathrm{map}(Y, K)_f, \mathrm{map}_*(Y, K)_f \in M_0(X))$$

(*where* $\mathrm{map}(Y, K)_f$, $\mathrm{map}_*(Y, K)_f$ *indicate the path components of f in the corresponding function spaces*).

In particular, for any connected space Y

$$M_0(X) \subset M(X \wedge Y).$$

(d) *If $F \to E \xrightarrow{f} B$ is a fibration and $B \in M(X)$ ($B \in M_0(X)$), then $F \in M(X)$ if and only if $E \in M(X)$ ($F \in M_0(X)$ if and only if $E \in M_0(X)$).*

An obvious special case: If $K \in M_0(X)$ then $\Omega K \in M(X)$.

(e) *If $X_0 \to X_1 \to X_2$ is a cofibration sequence, then*

$$M(X_0) \cap M(X_1) = M(X_0) \cap M(X_2),$$

$$M_0(X_0) \cap M_0(X_1) = M_0(X_0) \cap M_0(X_2).$$

If $X_1 \to X_2 \to \cdots X_n \cdots \to$ is a sequence of cofibrations, then $\bigcap_i M(X_i) \subset M(\lim_\to X_n)$ and $\bigcap_i M_0(X_i) \subset M_0(\lim_\to X_i)$.

(f) *If X is simply connected and $K \in M(\Omega X)$, then $B\mathrm{aut}\, K \in M(X)$.*

COROLLARY C′. *Let X be a CW complex satisfying*: (a) *$\pi_n(X)$ are locally finite*; (b) *$\pi_n(X) = 0$ for $n > N$. Then $\mathcal{K} \subset M(X)$ and $\mathcal{K}_0 \subset M_0(X)$ ($\mathcal{K}$, $\mathcal{K}_0$ as in* 1.3.1). *If X is simply connected, then $\mathcal{K}_0 \subset M(X)$ and if $K \in \mathcal{K}$, $B\mathrm{aut}\ K \subset M(X)$.*

In Section 4 we apply Theorem C and Corollary C′ to the theory of phantom maps:

THEOREM D. *Let X be a CW complex. Suppose $\pi_n(X) = 0$ for $n > N$. Let Y be a simply connected CW complex in $\mathcal{K}_0$ (see* 1.3.1). *Then*:

(i) $\tilde{\theta}(X, Y) \approx_w \mathrm{map}_*(X_Q, Y)$; *all path components of $\theta(X, Y)$ are homotopy equivalent and*

$$\pi_m(\tilde{\theta}(X, Y)) \approx \prod_n \operatorname{Ext}(H_n(X, Q), \pi_{n+m+1}(Y)/\text{torsion})$$

$$\approx \prod_n \operatorname{Hom}(H_n(X, Z)/\text{torsion}, \pi_{n+m+1}(Y)/\text{torsion}) \otimes \hat{Z}_{P_1}/Z_{P_1}.$$

(ii) *If Y is a rational H-space, then* $\tilde{\theta}(X, Y) \approx_w \Pi_m K(\pi_m \tilde{\theta}(X, Y), m)$.

(iii) *If $Y \in \mathcal{K}$ or if $Y \in \mathcal{K}_0$ and X is simply connected, then* $\tilde{\theta}(X, Y) \approx_w \operatorname{map}_*(X, Y)$.

(iv) *If X is 2-connected and K is a finite complex, then any fibration over X with fiber K is trivial over every skeleton of X (though may fail to be trivial).*

REMARK. For X just 1-connected, (iv) is obviously false: $S^{2n+1} \to CP^n \to CP^\infty$ is not trivial over the $2n+2$ skeleton of CP^∞.

In Section 5 we bring some examples of applications to Theorem D:

EXAMPLES E. (a) Consider the families of fibrations

$$(\xi)_{n,m}: S^n \to E \to K(Z, m), \qquad m > 2.$$

Then,

if n is odd, $(\xi)_{n,m}$ contains only the trivial fibration;

if $n = 2k$ and $m \neq 2k-1$ $(k > 1)$, $(\xi)_{n,m}$ contains only the trivial fibration;

if $n = 2k$ and $m = 2k-1$ $(k > 1)$, $(\xi)_{n,m}$ contains uncountably many equivalence classes of fibrations (all trivial when restricted to skeleta of $K(Z, 2k-1)$).

(b) Let K be a finite complex which is a rational H-space. Suppose $\pi_n K \otimes Q = 0$ for $n > n_0$. Given a prime p and an integer $n > n_0$ let $K_p^{(n)}$ be the p-localization of the $n-1$ connective fibering of K. Then $[K_p^{(n)}, K_p^{(n)}]$ is finite if $H^*(K, Q) = 0$ and uncountable otherwise.

(c) Let K be rationally acyclic finite complex. Then for any integer n, K is irreducible if and only if $K^{(n)}$ is irreducible. (K is irreducible if every map $h: L \to K$ $(L \neq *)$ which has a left homotopy inverse is a homotopy equivalence.)

2. Rationalization and phantom maps

First one observes that while studying phantom maps one may always assume that the target is simply connected: If $\tilde{Y} \to Y$ is a universal covering space then $\operatorname{map}_*(X, \tilde{Y}) \to \operatorname{map}_*(X, Y)$ is a homeomorphism onto the path components of $\operatorname{map}_*(X, Y)$ whose image in $\operatorname{Hom}(\pi_1(X), \pi_1(Y))$ is the null homomorphism. But as phantom maps are always trivial on fundamental groups $\tilde{\theta}(X, \tilde{Y}) \to \tilde{\theta}(X, Y)$ is a homeomorphism.

PROOF OF LEMMA A. (a) It is clear that Postnikov sections and connective fiberings of integral approximations are integral approximations. We construct the Postnikov tower of Y_0 inductively:

Suppose $j_n: P_n(Y) \to P_n(Y_0) = (P_n(Y))_0$ is an integral approximation of $P_n(Y)$ — the n-th stage Postnikov approximation of Y. Consider the next k-invariant:

$$\begin{array}{ccc} P_n(Y) & \xrightarrow{j_n} & P_n(Y_0) = (P_nY)_0 \\ \downarrow{\scriptstyle k_{n+2}} & & \downarrow{\scriptstyle (k_{n+2})_0} \\ K(\pi_{n+1}Y, n+2) & \xrightarrow{\lambda \operatorname{proj} = \lambda h} & K(\pi_{n+1}Y/\text{torsion}, n+2) \end{array}$$

As j_n is a rational equivalence some multiple of $h \circ k_{n+2}$ factors through $(P_nY)_0$. Replace h by that multiple, say λ, to ensure the existence of $(k_{n+2})_0$ so that $(k_{n+2})_0 \circ j_n \sim \lambda h \circ k_{n+2}$. Now looping $(k_{n+2})_0$ one obtains

$$\Omega P_n(Y_0) \xrightarrow{\Omega(k_{n+2})_0} K(\pi_{n+1}Y/\text{torsion}, n+1).$$

By the inductive hypothesis

$$\Omega P_n(Y_0) \approx \prod_{m \leqq n} K(\pi_m(Y)/\text{torsion}, m-1),$$

$$\operatorname{im} H^*(\Omega(k_{n+2})_0, Q) \subset PH^*(\Omega P_n(Y_0), Q) \subset QH^*(\Omega P_n(Y_0), Q).$$

The latter is zero in $\dim n+1$, hence $H^*(\Omega(k_{n+2})_0, Q) = 0$ and $\Omega(k_{n+2})_0$ has a finite order, say λ_1. Replace λh by $\lambda_1 \lambda h$ and $(k_{n+2})_0$ by $\lambda_1(k_{n+2})_0$. The latter will be the next k-invariant of Y_0 obtaining $P_{n+1}(Y_0)$.

(b) Given $h: X_1 \to X_2$ and $f: X_2 \to Y_0$, $H^*(h, Z_{P_1})$ injective and $f \circ h \sim *$. Suppose inductively one has a lifting

$$\begin{array}{ccccc} & & & \overset{f^{(n)}}{\nearrow} & Y_0^{(n)} \\ X_1 & \xrightarrow{h} & X_2 & \xrightarrow{f} & Y_0 \end{array}$$

where $r^{(n)}$ is the $n-1$ connective fibering over Y_0. Further assume inductively that $f^{(n)} \circ h \sim *$. Let $k^{(n)}: Y_0^{(n)} \to K(\pi_n(Y_0), n)$ be the next invariant whose fiber $\hat{r}^{(n+1)}: Y_0^{(n+1)} \to Y_0^{(n)}$ is the n-connective fibering. As $\pi_n(Y_0)$ is a free Z_{P_1} module and $H^*(h, Z_{P_1})$ is injective, $H^*(h, \pi_n(Y_0))$ is injective and $0 = [k^{(n)} \circ f^{(n)} \circ h] = H^*(h, \pi_n(Y_0))[k^{(n)} \circ f^{(n)}]$ implying $[k^{(n)} \circ f^{(n)}] = 0$ and $f^{(n)}$ lifts to $f^{(n+1)}: X_2 \to Y_0^{(n+1)}$. Now, $r^{(n+1)} \circ f^{(n+1)} \circ h \sim f^{(n)} \circ h \sim *$ implies that $f^{(n+1)} * h$ lifts to $X_1 \to K(\pi_n(Y_0), n-1) = \text{fiber } \hat{r}^{(n+1)}$. but $Y_0^{(n)}$ is an integral approximation and therefore

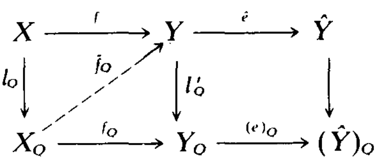
$$\Omega K^{(n)}\colon \Omega Y_0^{(n)} \approx \prod_{m=n}^{\infty} K(\pi_m(Y_0), m-1) \to K(\pi_n(Y_0), n-1)$$

is a projection and has a right inverse. Consequently $K(\pi_n(Y_0), n-1) \to Y_0^{(n+1)}$ is null homotopic, $f^{(n+1)} \circ h \sim *$ and one can proceed by induction to further lift $f^{(n+1)}\colon X_2 \to Y_0^{(n+1)}$ to higher connected fiberings. Hence $f \sim *$.

PROOF OF THEOREM B. (a) This is a consequence of 1.2(c) and Lemma A(b).

(b) (after Meier [10]). Let $\hat{e}\colon Y \to \hat{Y}$ be the Sullivan profinite completion which coincides with the Bousfield–Kan $\Pi_{p\in P}(F)_{P\to\infty} Y$. By a theorem of Sullivan $\hat{Y}$ cannot be a target of a non-trivial phantom map. It follows that if $f\colon X \to Y$ is a phantom map, $\hat{e} \circ f \sim *$. As Y is of finite type the converse holds as well: If $\hat{e} \circ f \sim *$ and $h\colon K \to X$ is a map of a finite polyhedra, $\hat{e} \circ f \circ h \sim *$. But by the Bousfield–Kan version of the arithmetic square ([2] p. 192, 8.1(ii)) $\hat{e}_*\colon [K, Y] \to [K, \hat{Y}]$ is injective, hence $f \circ h \sim *$ and f is phantom:

Now suppose $f\colon X \to Y$ satisfies $f \circ \tau \sim *$, or equivalently $f \sim \tilde{f}_Q \circ l_Q$, $\tilde{f}_Q\colon X_Q \to Y$. As $\mathrm{map}_*(X_Q, \hat{Y}) \approx_w \hat{e} \circ \tilde{f}_Q \sim *$, $\hat{e} \circ f \sim *$ and f is phantom. Conversely, suppose f is phantom. Using the same arithmetic square one has the following:



The right hand square of the diagram is a homotopy pull back. Now $\hat{e} \circ f \sim *$ implies $(\hat{e} \circ f)_Q = \hat{e}_Q \circ f_Q \sim *$, hence, f_Q lifts to $\tilde{f}_Q\colon X_Q \to Y$ ($\hat{e} \circ f_Q \sim *$ as $[X_Q, \hat{Y}] = *$). It follows that f and $\tilde{f}_Q \circ l_Q$ are homotopic after composing with $\hat{e}$ and with $l'_Q\colon Y \to Y_Q$. The obstruction for $\tilde{f}_Q \circ l_Q \sim f$ is a map $X \to \Omega(\hat{Y})_Q$. But any such map factors through $X \to X_Q \to \Omega(\hat{Y})_Q$ and one can use $X_Q \to \Omega(\hat{Y})_Q$ to alter the lifting $\tilde{f}_Q$ to obtain $\tilde{f}_Q \circ l_Q \sim f$.

(c) In diagram (DB) the homotopy fibers of τ^* and τ_0^* (over the constant maps) are $\mathrm{map}_*(X_Q, Y)$ and $\mathrm{map}_*(X_Q, Y_Q)$ respectively. As τ^* and τ_0^* in $(\mathrm{DB})_0$ are the restrictions of their corresponding fibrations in (DB) they have the same fibers and the map between these fibers, $j_*^Q\colon \mathrm{map}_*(X_Q, Y) \to \mathrm{map}_*(X_Q, Y_0)$, is a homotopy equivalence.

(d) By (c) $\tilde{\theta}(X, Y)$ is the homotopy pull-back of maps between path connected spaces. Thus $\pi_0(\tilde{\theta}(X, Y)) = \theta(X, Y)$ is in 1–1 correspondence with the double cosets in im $\pi_1(j_*^\tau)\backslash \pi_1(C_0(X_\tau, Y_0), *)/\mathrm{im}\, \pi_1(\tau_0^*)$. As

$$\pi_1(C_0(X_\tau, Y_0)) = [X_\tau, \Omega Y_0] = \Pi H^m(X_\tau, \pi_{m-1}(Y)/\mathrm{torsion})$$

is an abelian group

$$\theta(X, Y) = \pi_1(C_0(X_\tau, Y_0))/\text{im}\,\pi_1(j_*^\tau) + \text{im}\,\tau_1(\tau_0^*).$$

Now,

$$\text{coker}\,\pi_1(\tau_0^*) = \pi_1 C_0(X_\tau, Y_0)/\text{im}\,\pi_1(C_0(X, Y_0))$$

$$= \Pi H^m(X_\tau, \pi_{m-1}(Y)/\text{torsion})/H^m(X, \pi_{m-1}(Y)/\text{torsion})$$

(as by 1.2(c) $H^m(\tau, \pi_{m-1}(Y)/\text{torsion})$ is a monomorphism). This in turn is isomorphic to

$$\prod_n H^m(X_Q, \pi_m(Y)/\text{torsion}) \approx \prod_n \text{Ext}(H_{m-1}(X, Q), \pi_m(Y)/\text{torsion}).$$

$\theta(X, Y)$ is a quotient of this group by image $\{[X_\tau, \Omega Y] \to [X_\tau, \Omega Y_0] \to \text{coker}\,\pi_1(\tau_0^*)\}$.

3. Miller's "Sullivan conjecture"

In this section we observe that the range and domain of Miller's theorem 1.4.2 could be easily extended and prove Theorem C. For this we need:

3.1. LEMMA (e.g. see [12] 1.5). *Let φ: $K \to L$ be a simplicial map. Given a space X, denote by $V^\varphi(K, X)$ the subspace of* $\text{map}_*(K, X)$ *consisting of maps f: $K \to X$ so that $f \mid \varphi^{-1}(x) \sim *$ for every $x \in L$. If for every $x \in L$ $C_0(\varphi^{-1}(x), X) \approx_w *$ then the* map φ^*: $\text{map}_*(L, X) \to V^\varphi(K, X) \subset \text{map}_*(K, X)$ *is a homotopy equivalence.*

PROOF OF THEOREM C. (a) Choose a simplicial model for the fibration $\Omega X \to \mathscr{L}X \to X$, $\varphi^{-1}(x) \approx \Omega X$ ($\mathscr{L}X$ is the space of based paths in X). Suppose $Y \in M_0(\Omega X)$. Obviously $V^\varphi(\mathscr{L}X, Y) \approx \text{map}_*(\mathscr{L}X, Y) \approx *$. Using 3.1 one has $\text{map}_*(X, Y) \approx V^\varphi(\mathscr{L}X, Y) \approx *$ and $Y \in M(X)$.

(b) Again assume φ: $E \to B$ is simplicial and $\varphi^{-1}(b) \approx F$ for every $b \in B$. If $Y \in M_0(F)$ by 3.1 $V^\varphi(E, Y) \approx \text{map}_*(B, Y)$. $C_0(E, Y)$ is a path component of $V^\varphi(E, Y)$ and obviously $\varphi^*(C_0(B, Y)) \subset C_0(E, Y)$. The above equivalence restricted to path components yields $C_0(B, Y) \approx C_0(E, Y)$ and $Y \in M_0(B)$ if and only if $Y \in M_0(E)$.

If $Y \in M(F)$, $\text{map}_*(E, Y) \approx V^\varphi(E, Y)$ and $\text{map}_*(E, Y) \approx \text{map}_*(B, Y)$. Hence $Y \in M(B)$ if and only if $Y \in M(E)$.

(c) Consider first the case $K \in M(X)$. As one has a fibration

$$\text{map}_*(X, \text{map}_*(Y, K)_f) \to \text{map}_*(X, \text{map}(Y, K)_f) \to \text{map}_*(X, K) \approx_w *$$

the first two spaces are equivalent and it suffices to show that any one of them is contractible.

Now, $\mathrm{map}_*(X, K) \approx_w *$ is equivalent to the assertion that the evaluation map E_{x_0}: $\mathrm{map}(X, K) \to K$ is an equivalence. Now, one has a pull back diagram

$$\begin{array}{ccc} \mathrm{map}(X, \mathrm{map}(Y, K)_f) & \to & \mathrm{map}(X, \mathrm{map}(Y, K)) = \mathrm{map}(Y, \mathrm{map}(X, K)) \\ \downarrow & & \downarrow \\ \mathrm{map}(Y, K)_f & \longrightarrow & \mathrm{map}(Y, K) \end{array}$$

As the evaluation $E_{x_0} = \mathrm{map}(Y, E_{x_0})$ on the right is a homotopy equivalence, so is the one on the left and $\mathrm{map}_*(X, \mathrm{map}(Y, K)_f) \approx_w *$. Now suppose $K \in M_0(X)$, hence E_{x_0}: $\hat{C}_0(X, K) \xrightarrow{\approx} K$. Now,

$$\hat{C}_0(X, \mathrm{map}(Y, K)_f) = \mathrm{map}(Y, \hat{C}_0(X, K))_{\chi\circ f}$$

(χ: $K \to \hat{C}_0(X, K)$ the inclusion of constant maps),

$$C_0(X, \mathrm{map}_*(Y, K)_f) = \mathrm{map}_*(Y, y_0; \hat{C}_0(X, K), \chi f(y_0))_{\chi\circ f}.$$

Again one has a commutative diagram

$$\begin{array}{ccc} \hat{C}_0(X, \mathrm{map}(Y, K))_f = \mathrm{map}(Y, \hat{C}_0(X, K))_{\chi\circ f} & \subset & \mathrm{map}(Y, \hat{C}_0(X, K)) \\ \downarrow & & \downarrow \\ \mathrm{map}(Y, K)_f & \subset & \mathrm{map}(Y, K) \end{array}$$

The left-hand-side vertical map is the restriction to a path component of the right-hand-side vertical homotopy equivalence, hence it is a homotopy equivalence and $C_0(X, \mathrm{map}(Y, K)_f) \approx_w *$.

Similarly for the pointed case:

As $\mathrm{map}_*(Y \wedge X, K) = \mathrm{map}_*(Y, C_0(X, K))$ if $K \in M_0(X) K \in M(Y \wedge X)$.

(d) One has a fibration

$$\begin{array}{ccc} \mathrm{map}_*(X, F) \to r_*^{-1} C_0(X, B) & \xrightarrow{r_*} & C_0(X, B) \\ \cap & & \\ \mathrm{map}_*(X, E) & & \end{array}$$

If $B \in M_0(X)$ $\mathrm{map}_*(X, F) \approx r_*^{-1} C_0(X, B)$. The restriction to path components yields $C_0(X, F) \xrightarrow{\approx} C_0(X, E)$ and $F \in M_0(X)$ if and only if $B \in M_0(X)$. If $B \in M(X)$, $r_*^{-1} C_0(X, B) = r_*^{-1} \mathrm{map}_*(X, B)$ and $\mathrm{map}_*(X, F) \approx \mathrm{map}_*(X, E)$; $F \in M(X)$ if and only if $E \in M(X)$.

(e) If $X_0 \xrightarrow{\sigma} X_1 \xrightarrow{\tau} X_2$ is a cofibration sequence,

$$\mathrm{map}_*(X_2, K) \xrightarrow{\tau^*} \mathrm{map}_*(X_1, K) \xrightarrow{\sigma^*} \mathrm{map}_*(X_0, K)$$

is a fibration in the sense that $\mathrm{map}_*(X_2, K)$ is the homotopy fiber of σ^* over the constant map, i.e.,

$$\mathrm{map}_*(X_2, K) \to \sigma^{*-1}(C_0(X_0, K)) \xrightarrow{\sigma^*} C_0(X_0, K)$$

is a fibration.

Now one proceeds as in the proof of (d): $K \in M_0(X_0)$ implies

$$\mathrm{map}_*(X_2 K) \approx \sigma^{*-1}(C_0(X_0, K))$$

which yields $C_0(X_2, K) \approx C_0(X_1, K)$ when restricted to path components. If $K \in M(X_0)$ then

$$\sigma^{*-1}(C_0(X_0, K)) = \mathrm{map}_*(X_1, K) \quad \text{and} \quad \mathrm{map}_*(X_2 K) \approx \mathrm{map}_*(X_1, K).$$

If $X_1 \to \cdots \to X_n \to \cdots$ a is sequence of cofibrations, $\mathrm{map}_*(X, K) = \lim_{\leftarrow} \mathrm{map}_*(X, K)$ where $X = \lim_{\to} X_n$. Moreover, $\mathrm{map}_*(X_n, K) \to \mathrm{map}_*(X_{n-1}, K)$ are fibrations. If $\mathrm{map}_*(X_n, K) \approx_w *$ for all n, $\mathrm{map}_*(X, K) \approx_w *$ as $\lim_{\leftarrow}$ and $\lim^1_{\leftarrow}$ of trivial groups are trivial. Thus $\bigcap_n M(X_n) \subset M(\lim_{\to} X_n)$.

Similarly for the M_0 case.

(f) One can reconstruct X as the classifying space of ΩX. $X = \lim_{\to} X_n$ where $X_1 = \Sigma \Omega X$, and for $n \geqq 1$ one has a cofibration

$$\underbrace{\Omega X * \Omega X * \cdots * \Omega X}_{n} \to X_{n-1} \to X_n,$$

where

$$\Omega X * \Omega X * \cdots * \Omega X \approx \Sigma^{n-1} \underbrace{\Omega X \wedge \cdots \wedge \Omega X}_{n}.$$

Hence one has a cofibration

$$X_{n-1} \to X_n \to \Sigma^{n} \underbrace{\Omega X \wedge \cdots \wedge \Omega X}_{n}.$$

If $K \in M(\Omega X)$, by (c) $K \in M(\Omega X \wedge \cdots \wedge \Omega X)$, $n \geqq 1$ and again by (c) $\mathrm{aut}_1 K = \mathrm{map}(K, K)_1 \in M(\Omega X \wedge \cdots \wedge \Omega X)$, $n \geqq 1$. As $\mathrm{map}_*(\Sigma Y, B(\mathrm{aut}_1 K)) \approx \mathrm{map}_*(Y, \mathrm{aut}_1 K)$, $\mathrm{aut}_1 K \in M(Y)$ if and only if $B\,\mathrm{aut}_1 K \in M(\Sigma Y)$, and for Y connected, $\mathrm{map}_*(\Sigma Y, B\,\mathrm{aut}_1 K) = \mathrm{map}_*(\Sigma Y, B\,\mathrm{aut}\, K)$. Hence, $\mathrm{aut}_1 K \in M(\Omega X \wedge \cdots \wedge \Omega X) \subset M(\Sigma^{n-1} \Omega X \wedge \cdots \wedge \Omega X)$ implies

$$B\operatorname{aut} K \in M\left(\sum^{n} \Omega X \wedge \cdots \wedge \Omega X\right) \qquad \text{for } n \geqq 1.$$

By (e) and by induction $B\operatorname{aut} K \in \cap M(X_n) \subset M(X)$.

PROOF OF COROLLARY C′. By 1.4.2, if G is locally finite and K is finite dimensional then $K \in M(BG) \subset M_0(BG)$. By Theorem C(a) one gets inductively $K \in M(K(\pi, n))$, $n \geqq 1$, a π-locally finite abelian group. For $n \geqq 2$ and for a space K whose loop space is finite dimensional $K \in M_0(BG)$, and then $K \in M(K(\pi, n))$. Using Theorem C(b) inductively one gets: For a space X with $\pi_n(X)$ locally finite, $\pi_n(X) = 0$ for $n > N$, and K finite dimensional, $K \in M(X)$. If ΩK is finite dimensional, $K \in M_0(X)$. If ΩK is finite dimensional and X simply connected, then $K \in M(X)$. The extensions to $\mathcal{H}$ and $\mathcal{H}_0$ follow from Theorem C(c) and (d) and the part about $B\operatorname{aut} K$ follows from (f).

4. Application to phantom maps

PROOF OF THEOREM D. (i) If $\pi_n(X) = 0$, $n > N$ then X_τ satisfies the hypothesis of Corollary C′, hence $C_0(X_\tau, Y) \approx_w *$. It follows that in Theorem B(c) and in diagram $(\mathrm{DB})_0$, $\tilde{\theta}(X, Y)$ is the homotopy fiber of τ_0^* over the constant map which is $\mathrm{map}_*(X_Q, Y) \approx_w \mathrm{map}_*(X_Q, Y_0)$. As τ_0^* in $(\mathrm{DB})_0$ is a map between path connected spaces, all the path components of its homotopy fiber are homotopy equivalent. The computations of $\pi_m \tilde{\theta}(X, Y) = \pi_m \mathrm{map}_*(X_Q, Y_0)$ for $n \geqq 0$ is straightforward:

$$\pi_m \mathrm{map}_*(X_Q, Y_0) = [X_Q, \Omega^m Y_0]$$

$$= \prod_k H^k(X_Q, \pi_{m+k}(Y_0)) = \prod_k H^k(X_Q, \pi_{m+k}(Y)/\text{torsion}).$$

The computation of $\pi_0 \tilde{\theta}(X, Y) = \theta(X, Y)$ follows the proof of Theorem B(d) as it was established there that

$$\theta(X, Y) = \operatorname{coker}\left\{[X_\tau, \Omega Y] \to [X_\tau, \Omega Y_0] \to \prod_m H^m(X_Q, \pi_m(Y)/\text{torsion})\right\}.$$

But here $[X_\tau, \Omega Y] = \pi_1(C_0(X_\tau, Y)) = 0$.

(ii) If Y is a rational H-space, one can choose $Y_0 = \Pi_m K(\pi_m(Y)/\text{torsion}, m)$ and then $\tilde{\theta}(X, Y) \approx \mathrm{map}_*(X_Q, Y_0)$ is a product of Eilenberg Maclane spaces.

(iii) Under both conditions X_τ, Y satisfy the hypothesis of Corollary C′, hence $* \approx \mathrm{map}_*(X_\tau, Y) \approx C_0(X_\tau, Y)$ and by Theorem B(b) $\mathrm{map}_*(X, Y) = (\tau^*)^{-1} \mathrm{map}_*(X_\tau, Y) = (\tau^*)^{-1} C_0(X_\tau Y) = \tilde{\theta}(X, Y)$.

(iv) Here X_τ is 1-connected: If $K \in \mathcal{K}$, by Corollary C′ $B\,\mathrm{aut}\,K \in M(X_\tau)$. If K is a finite complex $B\,\mathrm{aut}\,K$ is of finite type, thus any map $X \to B\,\mathrm{aut}\,K$ is a phantom map.

5. Some examples and applications

PROOF OF EXAMPLE E(a). $(\xi)_{n,m}$ are classified by maps $f\colon K(Z,m) \to B(\mathrm{aut}_1 S^n)$. By Theorem D(iv) $f_{n,m}$ is a phantom map and there is an isomorphism

$$\pi_0 \,\mathrm{map}_*(K(Z,m), B(\mathrm{aut}_1 S^n))$$

$$\approx \prod_k \mathrm{Hom}(H_k(K(Z,m),Z)/\mathrm{torsion}, \pi_{k+1}(B\,\mathrm{aut}_1 S^n)/\mathrm{torsion}) \otimes \hat{Z}/Z.$$

Denote this group by A_0. Now

$$\pi_j(B\,\mathrm{aut}_1 S^n) \otimes Q = \pi_{j-1}(\Omega^n S^n)_1 \otimes Q$$

$$= \pi_{n+j-1}(S^n) \otimes Q = \begin{cases} Q & \text{if } n = 2k, \quad j = 2k, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the above group A_0 is trivial unless $n = 2k$ and $H_{2k-1}(K(Z,m),Z) \otimes Q \neq 0$. This in turn happens only when $m = 2k-1$ and then $A_0 \approx \hat{Z}/Z$. To prove (b) and (c) one observes the following variant of Corollary C′:

5.1. LEMMA. *Let $\hat{K}$ be a profinite completion of a finite complex. Then*:

(a) *If X is simply connected finite type and $\pi_n X = 0$ for $n > N$ then* $\mathrm{map}_*(X, \hat{K}) \approx_w *$.

(b) *If X is 2-connected finite type CW complex then* $\mathrm{map}_*(X^{(n)}, \hat{K}^{(n)}) \approx_w \mathrm{map}_*(X, \hat{K})$ ($X^{(n)}$ *— the $n-1$ connective fibering*).

PROOF. (a) X_τ is connected, $* \approx_w \mathrm{map}_*(X_\tau, K) \approx_w \mathrm{map}_*(X_\tau, \hat{K})$. As $\tau\colon X_\tau \to X$ yields an isomorphism on $H^*(\ ,N)$, where N is any finite group of coefficients, $\mathrm{map}_*(X_\tau, \hat{K}) \overset{\approx}{\to} \mathrm{map}_*(X, \hat{K})$.

(b) One has a fibration $\Omega X_{n-1} \to X^{(n)} \overset{\varphi}{\to} X$. By (a) $\mathrm{map}_*(\Omega X_{n-1}, \hat{K}) \approx_w *$, thus by 3.1 $\mathrm{map}_*(X, \hat{K}) \overset{\approx}{\to} \mathrm{map}_*(X^{(n)}, \hat{K}) \approx V^\varphi(X^{(n)}, \hat{K})$. Now, $\mathrm{map}_*(X^{(n)}, \hat{K}^{(n)}) \overset{\approx}{\to} \mathrm{map}_*(X^{(n)}, \hat{K})$ always holds.

PROOF OF EXAMPLE E(b). If $H_*(K,Q) = 0$, $[K, \hat{K}_p]$ are finite, and by 5.1(b) $[K, \hat{K}_p] = [K^{(n)}, \hat{K}_p^{(n)}] = [K_p^{(n)}, K_p^{(n)}]$. If $H_*(K,Q) \neq 0$ one has rational equivalences $\alpha\colon \tilde{K} \to \Pi_i S^{2n_i+1}$, $\rho\colon \Pi \hat{S}_p^{2n_i+1} \to \hat{K}_p$. $[\Pi S^{2n_i+1}, \Pi \hat{S}_p^{2n_i+1}]$ is obviously uncountable and the assignment $f \to \rho \cdot f \circ \alpha$ is injective, hence $[K, \hat{K}_p]$ is uncountable and again by 5.1(b) so is $[K^{(n)}, \hat{K}_p^{(n)}] = [K_p^{(n)}, K_p^{(n)}]$.

PROOF OF EXAMPLE E(c). One may assume that $K = K_p$ for some p. For a 1-connected CW complex K with $\pi_1 K$ a finite p-group to be irreducible is equivalent to the property: Every $f\colon K \to K$ is either a homotopy equivalence or "pronilpotent": $\pi_i(f)$ is nilpotent for every i.

If K is reducible, one has

$$L \xrightarrow{h_2} K \xrightarrow{h_1} L,$$

$h_1 \circ h_2$ is a homotopy equivalence, $L \not\simeq k^*$ and $h = h_2 \circ h_1$ is not a homotopy equivalence. $h^{(n)}\colon K^{(n)} \to K^{(n)}$ is the image of h under the natural map $[K, K] \to [K^{(n)}, K^{(n)}]$ which is a bijection by 5.1(b). Hence $h^{(n)}$ is not an equivalence (for its inverse will have to be of the form $g^{(n)}g\colon K \to K$ and $g \circ h \approx 1 \sim h \circ g$). Obviously $h^{(n)}$ is not pronilpotent as $\pi_i(L) \neq 0$ for arbitrarily high degrees and $\pi_i(L^{(n)}) \subset \pi_i K^{(n)}$ is stable under $\pi_i(h^{(n)})^N$. Now, the bijection $[K, K] \to [K^{(n)}, K^{(n)}]$ sends equivalences to equivalences and nilpotents (which coincide with pronilpotent self-maps for a 1-connected finite complex K) to nilpotents. Hence if K is irreducible $[K, K]$ consists only of equivalences and nilpotents; so does $[K^{(n)}, K^{(n)}]$ and $K^{(n)}$ is irreducible.

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